to Group b on the basis of base pressure measurements. These show base pressures substantially different from what would be expected[8] if the boundary layer at the separation point were fully turbulent. On the other hand, flows F1, F2, F3, C1, C5, C6, and C7 are classed in Group c, because the measured base pressures are in agreement with the turbulent correlation.[8] It must be noted that if the boat-tail length L (i.e., distance from expansion corner to base) is sufficiently large, a boundary layer that underwent severe distortion at the corner may recover (e.g., by retransition to turbulence) before it reaches the base. In Sternberg's experiments, retransition occurred at a distance of about 20 δ_o from the corner, but there are not enough data on boat-tailed bases to judge how much L should be for boundary-layer distortion effects to disappear. Presumably this will depend on Reynolds number and a variety of other factors. It is possible that the absence of a noticeable effect on base pressure in C6 and C7 ($\Delta p/\tau_o \simeq 50$ to 70, $L/\delta_o \gtrsim 7$) is partly due to the length of the boat-tail surface, but no definite conclusions are warranted at this stage.

IV. Conclusion

It is interesting to see from Fig. 1 that all points above the line $\Delta p/\tau_o = 75$ belong to Groups a and b,‡ and all those below the line $\Delta p/\tau_o = 60$ belong to Group c. Between these two lines are a few points from both Groups b and c. Considering the uncertainties in estimating τ_o, there is a remarkable consistency among the experimental data in indicating that the parameter $\Delta p/\tau_o$ governs the occurrence of reversion, which is likely whenever Δp is more than about 70 τ_o.

In Figs. 1 and 2, we have not plotted the experimental results (of reverting flows) on cooled blunted cone-cylinder models reported in Ref. 7, because of the difficulty of estimating τ_o. Using the curves of Tetervin at the values of M_o, R_{θ_o} and temperature ratio quoted in Ref. 7, however, we obtain $\Delta p/\tau_o \simeq 50$ in cases (i) and (ii) of Ref. 7 in which laminar solutions for heat transfer agreed with the measurements downstream of the corner, and $\Delta p/\tau_o \simeq 3$ in case (iii), where the turbulent heat transfer solution was closer to the measurements. Considering the likely errors in estimating c_{f_o} on cooled walls, it appears that the parameter $\Delta p/\tau_o$ can be used to predict the occurrence of reversion by expansion on cooled surfaces also.

Figure 2 shows the same data in terms of c_p and c_{f_o}. (Note that $\Delta p/\tau_o = -c_p/c_{f_o}$.) Now c_{f_o} varies only by a factor of about 2 between all the different experiments, whereas τ_o varies by a factor of 7 as shown in Fig. 1. If the variation of c_{f_o} can be ignored, we may use the crude thumb rule that the flow may be expected to revert if $-c_p \gtrsim 0.2$. Estimating c_p by supersonic linear theory, this rule takes the form $\varepsilon \gtrsim 5.74\,(M_o{}^2-1)^{1/2}$ deg for reversion, and should be useful for Mach numbers less than about 3 and R_{θ_o} less than about 10^5. Its validity outside this range remains to be tested.

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‡ Morkovin's flow has $\Delta p/\tau_o = 160$ using the value of c_{f_o} quoted by him. This value (obtained from momentum integral balance) is rather low for the M_o, R_{θ_o} of the experiment, as Morkovin himself notes. Equation (1) gives a value about 50% higher, but even this gives $\Delta p/\tau_o \simeq 100$, well above the critical region of Fig. 1.

The Auxiliary Problem for Feasible Directions

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Introduction

FEASIBLE direction methods provide an effective means of obtaining solutions to nonlinear, constrained optimization problems. Of these methods, one of the most widely used is that of Ref. 1. Zoutendijk showed that a sequence of improvements to the objective function can be generated by the careful selection of a direction vector $\bar{S}$ at constraint boundaries, followed by a one-dimensional search in this direction until the next boundary is reached. The determination of the $\bar{S}$ vector is a subproblem which is usually solved for structural applications as a linear programing problem in an n-dimensional design space using the simplex algorithm. An alternative approach formulates the subproblem with a single quadratic constraint, which in turn can be cast as a special linear problem and solved directly. This Note compares the computational efficiency of these two means of determining the $\bar{S}$ vector and demonstrates the advantages of using the alternative approach.

Problem Statement

The direction vector is computed as the solution to the following auxiliary extremum problem, for which β is maximized subject to the conditions[1–3]

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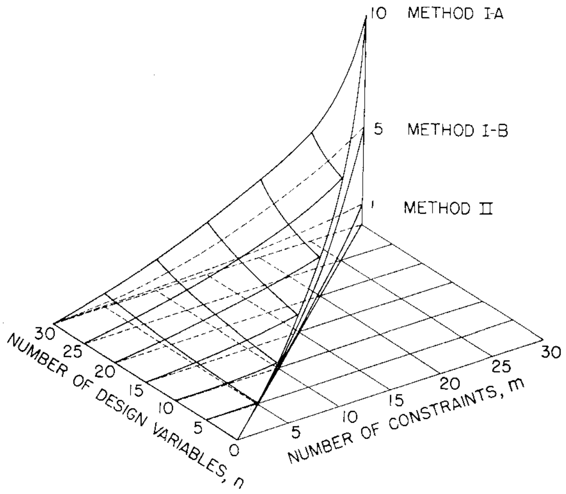

Fig. 1 Comparison of run times.

$$\bar{\nabla} g_j{}^T \bar{\mathbf{S}} + \theta_j \beta \leqq 0 \qquad j \in J \tag{1}$$

$$\bar{\nabla}^T F \cdot \bar{\mathbf{S}} + \beta \leqq 0 \tag{2}$$

$$|\bar{\mathbf{S}}| \text{ bounded} \tag{3}$$

where β = a scalar parameter; $\bar{\mathbf{S}}$ = the desired search vector whose elements are the variables of the auxiliary problem; $\bar{\nabla} g_j$ = the gradient of the jth constraint function; J = the set of constraints active at the design point in question; θ_j = a nonnegative scalar which is used to adjust the proximity of the search vector to the jth constraint boundary; $\bar{\nabla} F$ = the gradient of the objective function. Each solution of this problem then produces the optimal direction of search for design improvement, as indicated by the orientation of $\bar{\mathbf{S}}$, and a maximized value of β, which is a measure of satisfaction of the Kuhn-Tucker conditions for local optimality.[2]

The means of imposing the length restriction of condition (3) is arbitrary. The most common method is use of the infinite Cartesian norm (linear)

$$\text{Norm I:} \quad -1 \leqq s_i \leqq 1 \qquad i = 1, \ldots, n \tag{4}$$

where n is the number of design variables and the s_i are elements of $\bar{\mathbf{S}}$. A second and perhaps more attractive technique is to impose the Euclidian norm (quadratic)

$$\text{Norm II:} \quad \bar{\mathbf{S}}^T \bar{\mathbf{S}} \leqq 1 \tag{5}$$

This Technical Note compares, on a numerical basis, two methods of solving the auxiliary problem using Norm I with a third method using Norm II.

Linear Normalization

If Norm I is used, the auxiliary problem can be solved as a linear programing problem with the standard simplex algorithm[2] by applying the transformation

$$s_i' = s_i + 1 \qquad i = 1, \ldots, n \tag{6}$$

with n additional constraints

$$s_i' \leqq 2 \tag{7}$$

This approach shall be called method IA. An alternative method, termed IB, is to treat this linearized problem with a modified simplex procedure for upper bound constraints,[3] thus avoiding the additional constraints of Eq. (7).

Both methods thus feature conceptual simplicity and ease of application stemming from the widespread use of the simplex algorithm. However, these linear formulations can be criticized for their tendency to bias the solution for $\bar{\mathbf{S}}$ to the corners of the $|s_i| \leqq 1$ hypercube.[2] In particular, when the set J includes linear constraints, setting the appropriate θ_j's to zero will not ensure the tangency of $\bar{\mathbf{S}}$ components to these hyperplanes, an obvious requirement for maximal decrease in the objective.

Quadratic Normalization

The quadratic Norm II exhibits no bias because its limiting region is a unit hypersphere. Its use has been shown to be effective in optimal structural design.[4]

Zoutendijk[1] emphasized that Method II leads to more rapid convergence of nonlinear optimization problems than Method I because of the absence of this bias. Unfortunately, the complexity of the auxiliary problem appears to be increased when Norm II is used; however, this formulation can also be linearized by direct application of the Kuhn-Tucker conditions, the details of which are specified in Refs. 1 and 4.

Since an initial basic feasible solution to the linearized problems is always known a priori, computer codes can be written for each of the methods to minimize storage requirements. However, Method II usually requires less storage than the other methods. In addition, it demands less programing effort than either simplex algorithm and results in a corresponding reduction in total core requirements.

Numerical Efficiency

The computational effectiveness of methods IA and IB compared with Method II has been in question for some time,[3] and simplex iterations themselves have been known to consume considerable computational effort in practical applications when moderately large numbers of active constraints are involved.[5]

To assess the computational efficiency of each method, the auxiliary problem was solved for various combinations of the n design variables and m constraints, avoiding the condition of overconstraint, $m > n$. For a particular m and n pair, a set of gradient vectors of the objective and constraint functions of Eqs. (1) and (2) were generated randomly. Each of the three methods was then applied to the solution of this auxiliary problem, and the resulting computational times were recorded.

The elements of each vector were computed as uniformly distributed random numbers between ± 1.0, using the power residue method.[6] The starting value for each random vector generation was obtained from the instantaneous time-of-day calculation requested from the computing system. A meaningful $\bar{\mathbf{S}}$ was guaranteed through the requirement that each constraint gradient satisfy

$$\bar{\nabla} F^T \bar{\nabla} g_j \leqq 0 \tag{8}$$

The clock time included all the requirements for preparation and solution by each method, and three runs were made for each m and n combination so that an average could be used.

NUMBER OF DESIGN VARIABLES, n	Method	m = 5	m = 10	m = 15	m = 20	m = 25	m = 30
5	IA	0.13					
	IB	0.07					
	II	0.04					
10	IA	0.29	0.83				
	IB	0.14	0.45				
	II	0.05	0.16				
15	IA	0.81	1.72	3.11			
	IB	0.26	0.86	1.96			
	II	0.06	0.21	0.49			
20	IA	1.11	2.87	5.22	7.15		
	IB	0.31	1.34	2.74	4.02		
	II	0.07	0.24	0.59	1.06		
25	IA	1.68	3.50	6.19	9.80	15.84	
	IB	0.41	1.37	3.31	4.92	9.55	
	II	0.08	0.27	0.63	1.20	1.95	
30	IA	2.48	5.88	9.92	14.38	19.90	38.58
	IB	0.59	1.91	4.28	8.10	12.46	17.83
	II	0.09	0.30	0.65	1.28	2.09	3.72

NUMBER OF ACTIVE CONSTRAINTS, m

Fig. 2 Average run times (CPU sec).

All calculations were performed by an IBM 360/67 with FORTRAN G compilation, and the computer code for the three methods was written by the same person.

Figure 1 is a three-dimensional representation of the results of this comparison, where the height above the m-n grid indicates the relative time required to obtain an $\bar{S}$ vector in CPU sec. Figure 2 is a quantitative presentation of the same data. The penalty in computation for using the simplex methods becomes even more apparent for larger problems. For example, with $m = n = 50$, Method IA required 160 sec and Method IB 80 sec, while Method II required only 15 sec.

Conclusions

We conclude that the quadratic Euclidian normalization is superior to the linear Cartesian normalization as a candidate for the length restriction (3) in the statement of the auxiliary problem. The method of solution using Norm II is clearly more efficient than the methods based on Norm I, and has the additional advantage of introducing no bias. It is particularly noteworthy that Method II run times are primarily a function of the number of active constraints only, while Methods IA and IB run times increase significantly with increase in either m or n.

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On Transient Cylindrical Surface Heat Flux Predicted from Interior Temperature Response

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I. Introduction

IN heat transfer studies surface temperature and heat flux are important quantities. However, many experimental difficulties arise in implanting a probe at the surface for heat transfer measurements; for example, involving the motion of a projectile over a barrel surface, sliding of a piston in the combustion chamber, melting or ablation of a heat shield, freezing or quenching of a material process, and high temperature exhaustion of a rocket engine. Furthermore, the presence of a probe at the surface disturbs the surface condition and flow process adjacent to it and thus the true heat transfer. Therefore, it is desirable in these cases that the prediction of surface temperature and heat flux be accomplished by inverting the temperature as measured by a probe located interior to the surface of the solid material.

In general, the preceding problem is known as the "Inverse Problem." Many configurations such as spheres, slabs, and cylinders have been studied, and many methods such as numerical, graphical, series, convolution integral, and Laplace transforms have been utilized. Stolz[1] and Beck[2] considered the numerical inversion of the integral solution for semi-infinite and spherical bodies. In this method, care is required in selecting a time interval in order to achieve a stable solution. Carslaw and Jaeger,[3] Burggraf,[4] Koveryanov,[5] and Shumakov,[6] respectively, considered different series approaches in which generally the local temperature and local heat flux at an interior location and their higher derivatives are required. Sparrow, Hadji-Sheikh, and Lundgren,[7] Imber and Kahn,[8] Imber,[9] Sabherwal,[10] Masket and Vastano,[11] and Deverall and Channapragada[12] applied the transform method. In these works, the solution is represented in either an integral form after some manipulation of the contour integral from the inverse transform, or as a series form after an expansion of the solution for small and large times. Beck,[13] in a series of papers, applied a finite-difference approximation in conjunction with a least-squares fit procedure as well as a nonlinear estimate method for the inverse heat conduction problem.

This paper reports a simple method of determining a short time transient surface temperature and heat flux for the case of a hollow cylinder based on the inversion of the temperature profile measured by only one interior probe.

II. Analysis

Consider a long hollow cylinder with sufficient wall thickness such that the outer surface temperature has a negligible response when the inner surface is exposed to a thermal pulse of a transient process. For example, in a gun barrel or combustion chamber the transient process requires a time duration in the order of milliseconds while the lag time of the outer surface for a significant response is in the order of seconds. This condition considerably simplifies the theoretical analysis as the outer boundary may be assumed to be infinite, and only one interior probe of the cylinder is required in the experimental measurement. The material of the cylinder is considered to be homogeneous and isotropic with constant thermal diffusivity, α. Let R_i and R_o be, respectively, the inner and outer surface radii, R_1 the radius of the probe location and t the dimensionless time. If the temperature of the cylinder is initially uniform at T_o, the mathematical problem governing the temperature T, may be written as

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r} \qquad \begin{matrix} 1 < r < r_o \simeq \infty \\ t > 0 \end{matrix} \tag{1}$$

$$\theta(r, 0) = 0 \tag{2}$$

$$\theta(\infty, t) = 0 \tag{3}$$

$$\theta(r_1, t) = f(t) \qquad 1 < r_1 < \infty \tag{4}$$

where $\theta = T - T_o$, $r = R/R_i$, $t = \alpha\tau/R_i^2$, and $f(t)$ is the interior temperature response of the thermocouple measured at $r = r_1$ at the dimensionless time t. The problem is to predict surface temperature $\theta(1, t)$ and heat flux per unit area

$$q = -(K/R_i)(\partial\theta/\partial r)|_{r=1} \tag{5}$$

where K is the thermal conductivity.

The problem can be solved by Laplace transformation. Let the transformation be

$$\bar{\theta}(r, s) = \int_0^\infty \theta\, e^{-ts}\, dt \tag{6}$$

When θ satisfies the Dirichlet's condition the temperature function θ is recovered by inversion of the Laplace transformation as

$$\theta(r, t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \bar{\theta}\, e^{st}\, ds \tag{7}$$

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